# THE ARENS-MICHAEL ENVELOPE OF A SMASH PRODUCT

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ABSTRACT. Given a Hopf algebra $H$ and an $H$-module algebra $A$, we explicitly describe the Arens-Michael envelope of the smash product $A \# H$ in terms of the Arens-Michael envelope of $H$ and a certain completion of $A$. We also give an example (Manin's quantum plane) showing that the result fails for non-Hopf bialgebras.

## 1. Introduction

The Arens-Michael envelope [5, 11] of an associative $\mathbb{C}$-algebra $A$ is the completion of $A$ with respect to the family of all submultiplicative seminorms on $A$. For example [12], the Arens-Michael envelope of the polynomial algebra $\mathbb{C}[t_1, \ldots, t_n]$ is the algebra of holomorphic functions on $\mathbb{C}^n$. More generally [9], if $A$ is the algebra of regular (i.e., polynomial) functions on a complex affine algebraic variety $V$, then the Arens-Michael envelope of $A$ is the algebra of holomorphic functions on $V$. This result suggests that the Arens-Michael envelope of a "quantized polynomial algebra" (see, e.g., [1,3]) can be viewed as a "quantized algebra of holomorphic functions". From this point of view, Arens-Michael envelopes can be potentially useful for the development of noncommutative complex analytic geometry. For further information on Arens-Michael envelopes, we refer to [7–9].

In this short note, we extend our earlier result obtained in [8]. Let $H$ be a Hopf algebra, let $A$ be an $H$-module algebra, and let $A \# H$ denote the smash product of $A$ by $H$. Assuming that $H$ is cocommutative, we proved [8, Theorem 2.2] that the Arens-Michael envelope of $A \# H$ is isomorphic to the analytic smash product of the "$H$-completion" of $A$ by the Arens-Michael envelope of $H$. Here our goal is to show that the result holds without the cocommutativity assumption, but fails for non-Hopf bialgebras.

## 2. Preliminaries

We shall work over the complex numbers $\mathbb{C}$. All associative algebras and algebra homomorphisms are assumed to be unital. Modules over algebras are also assumed to be unital (i.e., $1 \cdot x = x$ for each left $A$-module $X$ and for each $x \in X$).

By a topological algebra we mean a topological vector space $A$ together with the structure of an associative algebra such that the multiplication map $A \times A \to A$ is separately continuous. A complete, Hausdorff, locally convex topological algebra with jointly continuous multiplication is called a $\widehat{\otimes}$-*algebra* [4,11]. If $A$ is a $\widehat{\otimes}$-algebra, then the multiplication $A \times A \to A$ extends to a continuous linear map from the

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completed projective tensor product $A \mathbin{\widehat{\otimes}} A$ to $A$. In other words, a $\widehat{\otimes}$-algebra is just an algebra in the tensor category $(\mathbf{LCS}, \widehat{\otimes})$ of complete Hausdorff locally convex spaces. This observation can be used to define $\widehat{\otimes}$-coalgebras, $\widehat{\otimes}$-bialgebras, and Hopf $\widehat{\otimes}$-algebras; see, e.g., [2].

If $A$ is a $\widehat{\otimes}$-algebra, then a *left $A$-$\widehat{\otimes}$-module* is a complete, Hausdorff locally convex space $X$ together with the structure of a left $A$-module such that the action $A \times X \to X$ is jointly continuous. Right $A$-$\widehat{\otimes}$-modules and $A$-$\widehat{\otimes}$-bimodules are defined similarly.

Recall that a seminorm $\|\cdot\|$ on an algebra $A$ is *submultiplicative* if $\|ab\| \le \|a\|\|b\|$ for all $a, b \in A$. A topological algebra $A$ is said to be *locally $m$-convex* if its topology can be defined by a family of submultiplicative seminorms. Note that the multiplication in a locally $m$-convex algebra is jointly continuous. An *Arens-Michael algebra* is a complete, Hausdorff, locally $m$-convex algebra.

Let $A$ be a topological algebra. A pair $(\widehat{A}, \iota_A)$ consisting of an Arens-Michael algebra $\widehat{A}$ and a continuous homomorphism $\iota_A \colon A \to \widehat{A}$ is called *the Arens-Michael envelope* of $A$ [5, 11] if for each Arens-Michael algebra $B$ and for each continuous homomorphism $\varphi \colon A \to B$ there exists a unique continuous homomorphism $\widehat{\varphi} \colon \widehat{A} \to B$ making the following diagram commutative:

$$\begin{tikzcd} \widehat{A} \arrow[r, dashed, "\widehat{\varphi}"] & B \\ A \arrow[u, "\iota_A"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

The Arens-Michael envelope always exists and can be obtained as the completion[1] of $A$ with respect to the family of all continuous submultiplicative seminorms on $A$ (see [11] and [5, Chap. V]). This implies, in particular, that $\iota_A \colon A \to \widehat{A}$ has dense range. Clearly, the Arens-Michael envelope is unique in the obvious sense.

Each associative algebra $A$ becomes a topological algebra with respect to the strongest locally convex topology. The Arens-Michael envelope, $\widehat{A}$, of the resulting topological algebra will be referred to as the Arens-Michael envelope of $A$. That is, $\widehat{A}$ is the completion of $A$ with respect to the family of *all* submultiplicative seminorms.

If $H$ is a bialgebra (respectively, a Hopf algebra), then it is easy to show that $\widehat{H}$ is a $\widehat{\otimes}$-bialgebra (respectively, a Hopf $\widehat{\otimes}$-algebra) in a natural way (for details, see [7]).

In what follows, we will use standard notation from Hopf algebra theory. In particular, given a Hopf algebra $H$, the symbols $\mu_H$, $\Delta_H$, $\eta_H$, $\varepsilon_H$, $S_H$ will denote the multiplication, the comultiplication, the unit, the counit, and the antipode, respectively. We will often suppress the subscript "$H$", when no confusion is possible.

Let $H$ be a bialgebra. Recall that an *$H$-module algebra* is an algebra $A$ endowed with the structure of a left $H$-module such that the product $\mu_A \colon A \otimes A \to A$ and the unit map $\eta_A \colon \mathbb{C} \to A$ are $H$-module morphisms. For example, if $\mathfrak{g}$ is a Lie algebra acting on $A$ by derivations, then the action $\mathfrak{g} \times A \to A$ extends to a map $U(\mathfrak{g}) \times A \to A$ making $A$ into a $U(\mathfrak{g})$-module algebra. Similarly, if $G$ is a semigroup acting on $A$ by endomorphisms, then $A$ becomes a $\mathbb{C}G$-module algebra, where $\mathbb{C}G$ denotes the semigroup algebra of $G$.

[1]Here we follow the convention that the completion of a non-Hausdorff locally convex space $E$ is defined to be the completion of the associated Hausdorff space $E/\overline{\{0\}}$.

Given an $H$-module algebra $A$, the *smash product algebra* $A \# H$ is defined as follows (see, e.g., [10]). As a vector space, $A \# H$ is equal to $A \otimes H$. To define multiplication, denote by $\mu_{H,A} \colon H \otimes A \to A$ the action of $H$ on $A$, and define $\tau \colon H \otimes A \to A \otimes H$ as the composition

$$H \otimes A \xrightarrow{\Delta_H \otimes 1_A} H \otimes H \otimes A \xrightarrow{1_H \otimes c_{H,A}} H \otimes A \otimes H \xrightarrow{\mu_{H,A} \otimes 1_H} A \otimes H \tag{1}$$

(here $c_{H,A}$ denotes the flip $H \otimes A \to A \otimes H$). Using Sweedler's notation, we have

$$\tau(h \otimes a) = \sum_{(h)} h_{(1)} \cdot a \otimes h_{(2)}. \tag{2}$$

Then the map

$$(A \otimes H) \otimes (A \otimes H) \xrightarrow{1_A \otimes \tau \otimes 1_H} A \otimes A \otimes H \otimes H \xrightarrow{\mu_A \otimes \mu_H} A \otimes H \tag{3}$$

is an associative multiplication on $A \otimes H$. The resulting algebra is denoted by $A \# H$ and is called the *smash product* of $A$ with $H$. Using (2), we see that the multiplication on $A \# H$ is given by

$$(a \otimes h)(a' \otimes h') = \sum_{(h)} a(h_{(1)} \cdot a') \otimes h_{(2)} h'. \tag{4}$$

In particular, we have

$$(a \otimes 1)(a' \otimes h') = aa' \otimes h', \tag{5}$$

$$(a \otimes h)(1 \otimes h') = a \otimes hh', \tag{6}$$

$$(1 \otimes h)(a \otimes 1) = \tau(h \otimes a). \tag{7}$$

This implies that $A$ and $H$ become subalgebras of $A \# H$ via the maps $a \mapsto a \otimes 1$ and $h \mapsto 1 \otimes h$.

Similar definitions apply in the $\widehat{\otimes}$-algebra case. Namely, if $H$ is a $\widehat{\otimes}$-bialgebra, then an *$H$-$\widehat{\otimes}$-module algebra* is a $\widehat{\otimes}$-algebra $A$ together with the structure of a left $H$-$\widehat{\otimes}$-module such that the product $A \widehat{\otimes} A \to A$ and the unit map $\mathbb{C} \to A$ are $H$-module morphisms. By replacing $\otimes$ with $\widehat{\otimes}$ in (1) and (3), we obtain an associative, jointly continuous multiplication on $A \widehat{\otimes} H$. The resulting $\widehat{\otimes}$-algebra is denoted by $A \mathbin{\widehat{\#}} H$ and is called the *analytic smash product* of $A$ with $H$.

Let $H$ be an algebra, and let $A$ be a left $H$-module. We say that a seminorm $\|\cdot\|$ on $A$ is *$H$-stable* [8] if for each $h \in H$ there exists $C > 0$ such that $\|h \cdot a\| \le C\|a\|$ for each $a \in A$. If $H$ is a bialgebra and $A$ is an $H$-module algebra, then the *$H$-completion* of $A$ is the completion of $A$ with respect to the family of all $H$-stable, submultiplicative seminorms. The $H$-completion of $A$ will be denoted by $\widetilde{A}$. It is immediate from the definition that $\widetilde{A}$ is an Arens-Michael algebra.

**Proposition 1** ( [8, Proposition 2.1]). *Let $H$ be a bialgebra, and let $A$ be an $H$-module algebra. Then the action of $H$ on $A$ uniquely extends to an action of $\widehat{H}$ on $\widetilde{A}$, so that $\widetilde{A}$ becomes an $\widehat{H}$-$\widehat{\otimes}$-module algebra. Moreover, the smash product $\widetilde{A} \mathbin{\widehat{\#}} \widehat{H}$ is an Arens-Michael algebra.*

## 3. The results

**Theorem 2.** *Let $H$ be a Hopf algebra, and let $A$ be an $H$-module algebra. Then the canonical map $A \# H \to \widetilde{A} \mathbin{\widehat{\#}} \widehat{H}$ extends to a $\widehat{\otimes}$-algebra isomorphism*

$$(A \# H)^{\frown} \cong \widetilde{A} \mathbin{\widehat{\#}} \widehat{H}.$$

*Proof.* Let $\varphi\colon A \# H \to B$ be a homomorphism to an Arens-Michael algebra $B$. We endow $A$ and $H$ with the topologies inherited from $\widetilde{A}$ and $\widehat{H}$, respectively. Since the canonical image of $A \# H$ is dense in $\widetilde{A} \mathbin{\widehat{\#}} \widehat{H}$, it suffices to show that $\varphi$ is continuous with respect to the projective tensor product topology on $A \# H$.

Define $\varphi_1\colon A \to B$ and $\varphi_2\colon H \to B$ by $\varphi_1(a) = \varphi(a \otimes 1)$ and $\varphi_2(h) = \varphi(1 \otimes h)$. Clearly, $\varphi_1$ and $\varphi_2$ are algebra homomorphisms. Using (5), we have

$$\varphi(a \otimes h) = \varphi\bigl((a \otimes 1)(1 \otimes h)\bigr) = \varphi_1(a)\varphi_2(h)$$

for each $a \in A,\ h \in H$. Therefore we need only prove that $\varphi_1$ and $\varphi_2$ are continuous.

Let $\|\cdot\|$ be a continuous submultiplicative seminorm on $B$. Then the seminorms $a \mapsto \|a\|' = \|\varphi_1(a)\|$ $(a \in A)$ and $h \mapsto \|h\|'' = \|\varphi_2(h)\|$ $(h \in H)$ are submultiplicative. This implies, in particular, that $\varphi_2$ is continuous. To prove the continuity of $\varphi_1$, we have to show that $\|\cdot\|'$ is $H$-stable.

For each $h \in H,\ a \in A$ we have the following identities in $A \# H$:

$$\begin{aligned}
h \cdot a \otimes 1 &= \sum_{(h)} h_{(1)} \cdot a \otimes \varepsilon(h_{(2)})1 \\
&= \sum_{(h)} h_{(1)} \cdot a \otimes h_{(2)} S(h_{(3)}) \\
&= \sum_{(h)} \tau(h_{(1)} \otimes a)(1 \otimes S(h_{(2)})) && \text{by (2) and (6)} \\
&= \sum_{(h)} (1 \otimes h_{(1)})(a \otimes 1)(1 \otimes S(h_{(2)})) && \text{by (7).}
\end{aligned}$$

Therefore

$$\begin{aligned}
\|h \cdot a\|' = \|\varphi_1(h \cdot a)\| &= \|\varphi(h \cdot a \otimes 1)\| \\
&= \Bigl\| \sum_{(h)} \varphi_2(h_{(1)})\varphi_1(a)\varphi_2(S(h_{(2)})) \Bigr\| \\
&\le \sum_{(h)} \|h_{(1)}\|'' \|a\|' \|S(h_{(2)})\|'' = C\|a\|',
\end{aligned}$$

where $C = \sum_{(h)} \|h_{(1)}\|'' \|S(h_{(2)})\|''$. Thus $\|\cdot\|'$ is $H$-stable, and so $\varphi_1$ is continuous. In view of the above remarks, $\varphi$ is also continuous, and so it uniquely extends to a $\widehat{\otimes}$-algebra homomorphism $\widetilde{A} \mathbin{\widehat{\#}} \widehat{H} \to B$. This completes the proof. $\square$

*Example* 3.1. It is natural to ask whether Theorem 2 holds in the more general situation where $H$ is a bialgebra. The following example shows that the answer is negative. Let $A = \mathbb{C}[x]$ be the polynomial algebra, and let the additive semigroup $\mathbb{Z}_+$ act on $A$ by

$$(k \cdot f)(x) = f(q^{-k}x) \quad (f \in \mathbb{C}[x],\ k \in \mathbb{Z}_+),$$

where $q \in \mathbb{C} \setminus \{0\}$ is a fixed constant. Then $A$ becomes an $H$-module algebra, where $H = \mathbb{C}\mathbb{Z}_+$ is the semigroup algebra of $\mathbb{Z}_+$.

Given $k \in \mathbb{Z}_+$, let us write $\delta_k$ for the corresponding element of $H$. If we identify $H$ with the polynomial algebra $\mathbb{C}[y]$ by sending the generator $\delta_1 \in H$ to $y$, then we obtain a vector space isomorphism $A \# H \cong \mathbb{C}[x, y]$. A straightforward computation shows that the resulting multiplication on $\mathbb{C}[x, y]$ is given by the formula $xy = qyx$. In other words, we can identify $A \# H$ with Manin's quantum plane [6]

$$\mathbb{C}_q[x, y] = \mathbb{C}\big\langle x, y \,\big|\, xy = qyx \big\rangle.$$

Suppose now that $|q| < 1$, and let $\|\cdot\|$ be an $H$-stable seminorm on $A$. Then there exists $C > 0$ such that $\|\delta_1 \cdot f\| \le C\|f\|$ for all $f \in A$. Setting $f = x^n$ and using the relation $\delta_1 \cdot x^n = q^{-n}x^n$, we see that

$$|q|^{-n}\|x^n\| \le C\|x^n\| \quad (n \in \mathbb{Z}_+).$$

Since $|q| < 1$, we conclude that there exists $N \in \mathbb{N}$ such that $\|x^n\| = 0$ for $n > N$.

It is easy to see that each seminorm of the form

$$\|a\|_N = \sum_{i=0}^{N} |c_i| \quad \Big(a = \sum c_i x^i \in A\Big)$$

is submultiplicative and $H$-stable. Moreover, it follows from the above remarks that each $H$-stable seminorm on $A$ is dominated by $\|\cdot\|_N$ for some $N$. Therefore the $H$-completion $\widetilde{A}$ is the completion of $A$ with respect to the topology generated by the seminorms $\|\cdot\|_N$, $N \in \mathbb{N}$. Thus $\widetilde{A}$ can be identified with the algebra $\mathbb{C}[[x]]$ of formal power series endowed with the topology of coordinatewise convergence. Since $\widehat{H}$ is isomorphic to the algebra of entire functions $\mathscr{O}(\mathbb{C})$ [12], we can identify the underlying topological vector space of $\widetilde{A} \mathbin{\widehat{\#}} \widehat{H}$ with

$$\mathbb{C}[[x]] \mathbin{\widehat{\otimes}} \mathscr{O}(\mathbb{C}) \cong \Big\{ a = \sum_{i,j \in \mathbb{Z}_+} c_{ij} x^i y^j : \|a\|_{\rho,N} = \sum_{i=0}^{N} \sum_{j=0}^{\infty} |c_{ij}| \rho^j < \infty \ \forall \rho > 0 \Big\}. \tag{8}$$

On the other hand (see [9, Corollary 5.14]), the Arens-Michael envelope of the quantum plane $\mathbb{C}_q[x, y]$ (where $|q| < 1$) can be identified with

$$\Big\{ a = \sum_{i,j \in \mathbb{Z}_+} c_{ij} x^i y^j : \|a\|_\rho = \sum_{i,j=0}^{\infty} |c_{ij}| |q|^{ij} \rho^{i+j} < \infty \ \forall \rho > 0 \Big\}.$$

Comparing this with (8), we see that the canonical map $(A \# H)\widehat{\ } \to \widetilde{A} \mathbin{\widehat{\#}} \widehat{H}$ (which always exists by the very definition of the Arens-Michael envelope and by Proposition 1) is not onto. Thus Theorem 2 cannot be generalized to non-Hopf bialgebras.

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